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Spectral theory of some degenerate elliptic operators with local singularities

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ABSTRACT

This paper is based on our previous results (Haroske and Skrzypczak (2008) [23], Haroske and Skrzypczak (in press) [25]) on compact embeddings of Muckenhoupt weighted function spaces of Besov and Triebel–Lizorkin type with example weights of polynomial growth near infinity and near some local singularity. Our approach also extends (Haroske and Triebel (1994) [21]) in various ways. We obtain eigenvalue estimates of degenerate pseudodifferential operators of type $b_2 \circ p(x,\mathrm{D}) \circ b_1$ where $b_i \in L_{r_i}(\mathbb{R}^n, w_i)$, $w_i \in \mathcal{A}_\infty$, $i = 1, 2$, and $p(x,\mathrm{D}) \in \Psi^{-\varkappa}_{1,0}$, $\varkappa > 0$. Finally we deal with the 'negative spectrum' of some operator $H_\gamma = A - \gamma V$ for $\gamma \to \infty$, where the potential V may have singularities (in terms of Muckenhoupt weights), and A is a positive elliptic pseudodifferential operator of order $\varkappa > 0$, self-adjoint in $L_2(\mathbb{R}^n)$. This part essentially relies on the Birman–Schwinger principle. We conclude this paper with a number of examples, also comparing our results with preceding ones.

1. Introduction

In recent years, some attention has been paid to compactness of embeddings of function spaces of Besov and Sobolev type as well as to analytic and geometric quantities describing this compactness, in particular, corresponding approximation and entropy numbers. In [16] Edmunds and Triebel proposed a program to investigate the spectral properties of certain pseudodifferential operators based on the asymptotic behaviour of entropy and approximation numbers, together with Carl's inequality and the Birman–Schwinger principle. Similar questions in the context of weighted function spaces of this type were studied by the first named author and Triebel, cf. [20,21,26], and were continued and extended by Kühn, Leopold, Sickel and the second author in [29–31]. All these papers were devoted to the class of so-called 'admissible' weights: These are smooth weights with no singular points, with $w(x) = (1+|x|^2)^{\alpha/2}$, $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$, as a prominent example.

We dealt in [23–25] with a different approach and considered weights from the Muckenhoupt class $\mathcal{A}_\infty$ which – unlike 'admissible' weights – may have local singularities, that can influence embedding properties of such function spaces. Weighted Besov and Triebel–Lizorkin spaces with Muckenhoupt weights are well-known concepts, cf. [7–10], and, more recently, [5,6,18,36,22]. In particular, we considered in [25] the weight

$$w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x) = \begin{cases} |x|^{\alpha_1}(1-\log|x|)^{\alpha_2} & \text{if } |x| \leqslant 1, \\ |x|^{\beta_1}(1+\log|x|)^{\beta_2} & \text{if } |x| > 1, \end{cases} \tag{1.1}$$

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where

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2), \quad \alpha_1 > -n,\ \alpha_2 \in \mathbb{R}, \qquad \boldsymbol{\beta} = (\beta_1, \beta_2), \quad \beta_1 > -n,\ \beta_2 \in \mathbb{R}. \tag{1.2}$$

This covers and partially extends our results [23] related to weights of purely polynomial growth both near the origin and for $|x| \to \infty$,

$$w_{\alpha,\beta}(x) \sim \begin{cases} |x|^{\alpha} & \text{if } |x| \leqslant 1, \\ |x|^{\beta} & \text{if } |x| > 1, \end{cases} \quad \text{with } \alpha > -n,\ \beta > -n. \tag{1.3}$$

We obtained wavelet decompositions and sharp criteria for the continuity or compactness of embeddings of type

$$\mathrm{id} : A^{s_1}_{p_1,q_1}\big(\mathbb{R}^n, w\big) \hookrightarrow A^{s_2}_{p_2,q_2}\big(\mathbb{R}^n\big),$$

where $s_2 \leqslant s_1$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leqslant \infty$, and $A^s_{p,q}$ stands for either Besov spaces $B^s_{p,q}$ or Triebel–Lizorkin spaces $F^s_{p,q}$. Moreover, for $w = w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ given by (1.1) we determined in [23,25] the exact asymptotic behaviour of corresponding entropy and approximation numbers, e.g.,

$$e_k\big(\mathrm{id} : A^{s_1}_{p_1,q_1}\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big) \hookrightarrow A^{s_2}_{p_2,q_2}\big(\mathbb{R}^n\big)\big) \sim k^{-\mu}, \quad k \in \mathbb{N},$$

where $\mu > 0$ depends on the parameters of the weight and the function spaces, as usual.

It is our main intention now to present some typical application of our previous results [23,25] to estimate the distribution of eigenvalues of degenerate pseudodifferential operators of type

$$B = b_2(x) p(x, \mathrm{D}) b_1(x), \tag{1.4}$$

where b_1, b_2 are (singular) functions belonging to some weighted L_p spaces of the above type, and $p(x, \mathrm{D})$ is in the Hörmander class $\Psi^{\varkappa}_{1,0}$ with $\varkappa < 0$. We obtain (nearly) sharp results as far as the expected order of the eigenvalue distribution is concerned. In a wider sense the present approach can also be seen as an extension and continuation of [21] since we follow similar ideas and tackle related problems. We shall restrict ourselves to some typical situations in order to demonstrate the strength of our method, but we do not aim at completeness. Our first main result, Theorem 4.4, establishes that under assumptions of type $b_i w_i \in L_{r_i}(\mathbb{R}^n)$, $i = 1, 2$, with w_i of type (1.1), and $\varkappa > n(\frac{1}{r_1} + \frac{1}{r_2})$, $1 \leqslant r_1' < p < r_2 \leqslant \infty$, then

$$B = b_2(\mathrm{id} - \Delta)^{-\varkappa/2} b_1$$

is compact in $L_p(\mathbb{R}^n)$ with eigenvalue estimates of type

$$\big|\mu_k(B)\big| \leqslant c \big\| b_1 w_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \big| L_{r_1}\big(\mathbb{R}^n\big)\big\| \big\| b_2 w_{(\boldsymbol{\gamma},\boldsymbol{\eta})} \big| L_{r_2}\big(\mathbb{R}^n\big)\big\| k^{-\lambda}, \quad k \in \mathbb{N},$$

where the exponent λ depends on w_i, r_i and $\varkappa$. In particular, for $b_1 = b_2 = b$, $p = 2$, $2 < r \leqslant \infty$, $w_{(\boldsymbol{\alpha},\boldsymbol{\beta})} b \in L_r(\mathbb{R}^n)$ with

$$-n\left(\frac{1}{2} + \frac{1}{r}\right) < \min(\alpha_1, \beta_1) \leqslant \max(\alpha_1, \beta_1) < n\left(\frac{1}{2} - \frac{1}{r}\right), \quad \alpha_2, \beta_2 \in \mathbb{R},$$
$$\beta_1 > 0 \quad \text{or} \quad \beta_1 = 0 \quad \text{and} \quad \beta_2 > 0,$$

and

$$\frac{\varkappa}{2} - \frac{n}{r} > \max(\alpha_1, 0) \quad \text{or} \quad \frac{\varkappa}{2} - \frac{n}{r} = \alpha_1 > 0 \quad \text{and} \quad \alpha_2 > \frac{\varkappa}{n} - \frac{1}{r},$$

the eigenvalues of the compact operator $B = b(\mathrm{id} - \Delta)^{-\varkappa/2} b$ in $L_2(\mathbb{R}^n)$ can be estimated from above by

$$\big|\mu_k(B)\big| \leqslant c \big\| w_{(\boldsymbol{\alpha},\boldsymbol{\beta})} b \big| L_r\big(\mathbb{R}^n\big)\big\|^2 \begin{cases} k^{-\frac{2}{n}\beta_1 - \frac{2}{r}} (1 + \log k)^{-2\beta_2} & \text{if } 0 < \beta_1 < \frac{\varkappa}{2} - \frac{n}{r}, \\ k^{-\frac{\varkappa}{n}} & \text{if } \beta_1 > \frac{\varkappa}{2} - \frac{n}{r}, \text{ or } \beta_1 = \frac{\varkappa}{2} - \frac{n}{r} \text{ and } 2\beta_2 > \frac{\varkappa}{n}, \\ k^{-\frac{\varkappa}{n}} (1 + \log k)^{\frac{\varkappa}{n} - 2\beta_2} & \text{if } \beta_1 = \frac{\varkappa}{2} - \frac{n}{r} \text{ and } \beta_2 \leqslant 0, \\ k^{-2\beta_2} & \text{if } \beta_1 = 0 \text{ and } 0 < \beta_2 \leqslant \frac{1}{r}, \\ k^{-\frac{2}{r}} (1 + \log k)^{-2\beta_2 + \frac{2}{r}} & \text{if } \beta_1 = 0 \text{ and } \beta_2 > \frac{1}{r}. \end{cases}$$

As described in [21] in some detail, this result can also be considered in the larger context of integral operators in the widest sense of the word, since by Schwartz's kernel theorem (1.4) can be written, at least formally, as

$$Bf = \int B(x, y) f(y)\, \mathrm{d}y \tag{1.5}$$

with $B(x, y) = b_2(x) b(x, y) b_1(y)$, where $b \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ is C^∞ off the diagonal $x = y$. Hence singularities of the kernel $B(x, y)$ are connected with the functions b_1, b_2, and with the behaviour of b near the diagonal $x = y$. Though we are exclusively at the ΨDO-side in this paper, one should keep in mind this larger context of integral operators (in its extended meaning with distributional kernels). As for classical ΨDO-results we refer to [14,3,40,4,2], whereas comprehensive treatments about mapping properties and distributions of eigenvalues of integral operators can be found in [35,28].

As a second application we shall use our results together with the Birman–Schwinger principle to prove (nearly) sharp in order estimates for the negative spectrum of operators of type

$$H_\gamma = A - \gamma V(x) \quad \text{as } \gamma \to \infty, \tag{1.6}$$

in $L_2(\mathbb{R}^n)$, where A is assumed to be positive-definite and self-adjoint operator in $L_2(\mathbb{R}^n)$ of order $\varkappa > 0$, and $V(x)$ is a real (singular) function. Here our main result can be found in Theorem 5.1. For its special case of power type weights (1.3) this reads as follows. Let A and H_γ be as above, with $\varkappa > 0$, $w_{\alpha,\beta} V \in L_r(\mathbb{R}^n)$, $1 < r \leqslant \infty$, and assume that $-n(1+\frac{1}{r}) < \alpha < n(1-\frac{1}{r})$, $0 < \beta < n(1-\frac{1}{r})$, $\varkappa > \frac{n}{r} + \max(\alpha, 0)$, and $\varkappa \neq \beta + \frac{n}{r}$. Then one can estimate the negative part of the point spectrum by

$$\#\bigl\{\sigma_{\mathrm{p}}(H_\gamma) \cap (-\infty, 0]\bigr\} \leqslant c\bigl(\gamma \bigl\| V \bigl| L_r(w_{\alpha,\beta})\bigr\|^2\bigr)^{\frac{n}{\min(\varkappa,\beta+n/r)}} \quad \text{as } \gamma \to \infty.$$

The paper is organised as follows. In Section 2 we recall basic facts about Muckenhoupt weight classes and weighted function spaces needed later on. Section 3 is devoted to the continuity and compactness of the embeddings when dealing with $w = w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ given by (1.1) with (1.2) together with our results on the asymptotic behaviour of the entropy numbers. In Section 4 we study the eigenvalue distribution of degenerate elliptic operators, and Section 5 contains our results on the negative spectrum. We conclude the paper with a number of examples, referring also to the very classical problem to determine the number of negative eigenvalues of the hydrogen operator

$$H = -\Delta - \frac{c}{|x|}, \quad c > 0,$$

in $L_2(\mathbb{R}^3)$, i.e., to study $\#\{\sigma_{\mathrm{p}}(H) \cap (-\infty, -\varepsilon]\}$ as $\varepsilon \downarrow 0$.

2. Weighted function spaces

First we fix some notation. By $\mathbb{N}$ we denote the set of natural numbers, by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$, and by $\mathbb{Z}^n$ the set of all lattice points in $\mathbb{R}^n$ having integer components.

The positive part of a real function f is given by $f_+(x) = \max(f(x), 0)$. For two positive real sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ we mean by $a_k \sim b_k$ that there exist constants $c_1, c_2 > 0$ such that $c_1 a_k \leqslant b_k \leqslant c_2 a_k$ for all $k \in \mathbb{N}$; similarly for positive functions.

Given two (quasi-)Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.

All unimportant positive constants will be denoted by c, occasionally with subscripts. For convenience, let both $\mathrm{d}x$ and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure in the sequel. If not otherwise indicated, log is always taken with respect to base 2. As we shall always deal with function spaces on $\mathbb{R}^n$, we may often omit the '$\mathbb{R}^n$' from their notation for convenience.

2.1. Muckenhoupt weights

We briefly recall some fundamentals on Muckenhoupt classes $\mathcal{A}_p$. By a weight w we shall always mean a locally integrable function $w \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, positive a.e. in the sequel.

Definition 2.1. Let w be a weight function on $\mathbb{R}^n$ and $1 < p < \infty$. Then w belongs to the Muckenhoupt class $\mathcal{A}_p$, if there exists a constant $0 < A < \infty$ such that for all balls B the following inequality holds

$$\left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right)^{1/p}\left(\frac{1}{|B|}\int_B w(x)^{-p'/p}\,\mathrm{d}x\right)^{1/p'} \leqslant A, \tag{2.1}$$

where p' is the dual exponent to p given by $1/p' + 1/p = 1$ and $|B|$ stands for the Lebesgue measure of the ball B.

The limiting cases $p = 1$ and $p = \infty$ can be incorporated as follows. Let M stand for the Hardy–Littlewood maximal operator given by

$$Mf(x) = \sup_{B(x,r)\in\mathcal{B}} \frac{1}{|B(x,r)|}\int_{B(x,r)} \bigl|f(y)\bigr|\,\mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{2.2}$$

where $\mathcal{B}$ is the collection of all open balls

$$B(x,r)=\{y\in\mathbb{R}^n:\ |y-x|<r\},\quad r>0.$$

Definition 2.2. A weight w belongs to the Muckenhoupt class $\mathcal{A}_1$ if there exists a constant $0<A<\infty$ such that the inequality

$$Mw(x)\leqslant Aw(x)$$

holds for almost all $x\in\mathbb{R}^n$. The Muckenhoupt class $\mathcal{A}_\infty$ is given by

$$\mathcal{A}_\infty=\bigcup_{p>1}\mathcal{A}_p. \tag{2.3}$$

Since the pioneering work of Muckenhoupt [32–34], these classes of weight functions have been studied in great detail, we refer, in particular, to the monographs [19,42,45,41] for a complete account on the theory of Muckenhoupt weights. Let us only mention the important feature of decomposition of $\mathcal{A}_p$ weights into $\mathcal{A}_1$ weights based on the facts that for two $\mathcal{A}_1$ weights w_1, w_2, and $1\leqslant p<\infty$, then $w=w_1w_2^{1-p}\in\mathcal{A}_p$. Conversely, suppose that $w\in\mathcal{A}_p$, then there exist $v_1\in\mathcal{A}_1$, $v_2\in\mathcal{A}_1$ such that $w=v_1v_2^{1-p}$. Moreover, it is known that the minimum, maximum, and the sum of finitely many $\mathcal{A}_1$ weights yields again an $\mathcal{A}_1$ weight. We refer to the above-mentioned literature for proofs and further details. As usual, we use the abbreviation

$$w(\Omega)=\int_\Omega w(x)\,\mathrm{d}x, \tag{2.4}$$

where $\Omega\subset\mathbb{R}^n$ is some bounded, measurable set.

Example 2.3. We restrict ourselves to the examples used below. In [25] we studied the weight

$$w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x):=\begin{cases}|x|^{\alpha_1}(1-\log|x|)^{\alpha_2} & \text{if } |x|\leqslant 1,\\ |x|^{\beta_1}(1+\log|x|)^{\beta_2} & \text{if } |x|>1,\end{cases} \tag{2.5}$$

where

$$\boldsymbol{\alpha}=(\alpha_1,\alpha_2),\quad \alpha_1>-n,\ \alpha_2\in\mathbb{R},\qquad \boldsymbol{\beta}=(\beta_1,\beta_2),\quad \beta_1>-n,\ \beta_2\in\mathbb{R}. \tag{2.6}$$

This extends the special case studied in [23] corresponding to $\alpha_2=\beta_2=0$,

$$w_{\alpha_1,\beta_1}(x)=\begin{cases}|x|^{\alpha_1} & \text{if } |x|\leqslant 1,\\ |x|^{\beta_1} & \text{if } |x|>1,\end{cases} \tag{2.7}$$

with $\alpha_1>-n$, $\beta_1>-n$. As is well known, cf. [23,25,17], $w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\in\mathcal{A}_1$ for

$$\max(\alpha_1,\beta_1)\leqslant 0\quad\text{and}\quad \begin{cases}\alpha_2\geqslant 0 & \text{if } \alpha_1=0,\\ \beta_2\leqslant 0 & \text{if } \beta_1=0,\end{cases} \tag{2.8}$$

and $w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\in\mathcal{A}_r$, $1<r<\infty$, for $\max(\alpha_1,\beta_1)<n(r-1)$.

2.2. Function spaces of type $B^s_{p,q}(\mathbb{R}^n,w)$ and $F^s_{p,q}(\mathbb{R}^n,w)$ with $w\in\mathcal{A}_\infty$

Let $w\in\mathcal{A}_\infty$ be a Muckenhoupt weight, and $0<p<\infty$. Then the weighted Lebesgue space $L_p(\mathbb{R}^n,w)$ contains all measurable functions such that

$$\|f|L_p(\mathbb{R}^n,w)\|=\left(\int_{\mathbb{R}^n}|f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p} \tag{2.9}$$

is finite. Note that for $p=\infty$ one obtains the classical (unweighted) Lebesgue space,

$$L_\infty(\mathbb{R}^n,w)=L_\infty(\mathbb{R}^n),\quad w\in\mathcal{A}_\infty. \tag{2.10}$$

Thus we mainly restrict ourselves to $p<\infty$ in what follows.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and its dual $\mathcal{S}'(\mathbb{R}^n)$ of all complex-valued tempered distributions have their usual meaning here. Let $\varphi_0=\varphi\in\mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\varphi \subset \{y \in \mathbb{R}^n\colon\ |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \quad \text{if } |x| \leqslant 1, \tag{2.11}$$

and for each $j \in \mathbb{N}$ let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then $\{\varphi_j\}_{j=0}^{\infty}$ forms a *smooth dyadic resolution of unity*. Given any $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ its Fourier transform and its inverse Fourier transform, respectively. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then the compact support of $\varphi_j\mathcal{F}f$ implies by the Paley–Wiener–Schwartz theorem that $\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)$ is an entire analytic function on $\mathbb{R}^n$.

Definition 2.4. Let $0 < q \leqslant \infty$, $0 < p < \infty$, $s \in \mathbb{R}$ and $\{\varphi_j\}_j$ be a smooth dyadic resolution of unity. Assume $w \in \mathcal{A}_\infty$.

(i) *The weighted Besov space* $B^s_{p,q}(\mathbb{R}^n, w)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\| f \big| B^s_{p,q}(\mathbb{R}^n, w)\big\| = \big\| \big\{2^{js}\big\|\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)\big|L_p(\mathbb{R}^n, w)\big\|\big\}_{j\in\mathbb{N}_0}\big|\ell_q\big\| \tag{2.12}$$

is finite.

(ii) *The weighted Triebel–Lizorkin space* $F^s_{p,q}(\mathbb{R}^n, w)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\| f \big| F^s_{p,q}(\mathbb{R}^n, w)\big\| = \big\|\big\| \big\{2^{js}\big|\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)(\cdot)\big|\big\}_{j\in\mathbb{N}_0}\big|\ell_q\big\|\big|L_p(\mathbb{R}^n, w)\big\| \tag{2.13}$$

is finite.

Remark 2.5. The spaces $B^s_{p,q}(\mathbb{R}^n, w)$ and $F^s_{p,q}(\mathbb{R}^n, w)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_j$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces for $p, q \geqslant 1$), and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n, w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, similarly for the F-case, where the first embedding is dense if $q < \infty$; cf. [7]. Moreover, for $w_0 \equiv 1 \in \mathcal{A}_\infty$ we obtain the usual (unweighted) Besov and Triebel–Lizorkin spaces; we refer, in particular, to the series of monographs by Triebel [46,48–50] for a comprehensive treatment of the unweighted spaces. In the sequel we adopt the nowadays usual custom to write $A^s_{p,q}$ instead of $B^s_{p,q}$ or $F^s_{p,q}$, respectively, when both scales of spaces are meant simultaneously in some context (but always with the understanding of the same choice within one and the same embedding, if not otherwise stated explicitly).

The above spaces with weights of type $w \in \mathcal{A}_\infty$ have been studied systematically by Bui first in [7,8], with subsequent papers [9,10]. It turned out that many of the results from the unweighted situation have weighted counterparts: e.g., we have $F^0_{p,2}(\mathbb{R}^n, w) = h_p(\mathbb{R}^n, w)$, $0 < p < \infty$, where the latter are Hardy spaces, see [7, Theorem 1.4], and, in particular, $h_p(\mathbb{R}^n, w) = L_p(\mathbb{R}^n, w) = F^0_{p,2}(\mathbb{R}^n, w)$, $1 < p < \infty$, $w \in \mathcal{A}_p$, see [42, Chapter VI, Theorem 1]. Concerning (classical) Sobolev spaces $W^k_p(\mathbb{R}^n, w)$ built upon $L_p(\mathbb{R}^n, w)$ in the usual way, it holds

$$W^k_p(\mathbb{R}^n, w) = F^k_{p,2}(\mathbb{R}^n, w), \quad k \in \mathbb{N}_0,\ 1 < p < \infty,\ w \in \mathcal{A}_p, \tag{2.14}$$

cf. [7, Theorem 2.8]. Further results can be found in [7,8,19,36,18,5,6]. In [37] the above class of weights was extended to the class $\mathcal{A}^{\mathrm{loc}}_p$. We partly rely on our approaches [22–25].

2.3. Mapping properties

We collect results of Bui in [7] concerning the lift and (some) pseudodifferential operators acting in $A^s_{p,q}(\mathbb{R}^n, w)$, $w \in \mathcal{A}_\infty$. Let $\varkappa \in \mathbb{R}$ and $0 \leqslant \gamma \leqslant 1$, then the Hörmander class $S^\varkappa_{1,\gamma}$ consists of all complex C^∞ functions $p(x,\xi)$ in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\big|\mathrm{D}^\beta_x\mathrm{D}^\alpha_\xi p(x,\xi)\big| \leqslant c_{\alpha,\beta}\big(1 + |\xi|^2\big)^{\frac{1}{2}(\varkappa - |\alpha| + \gamma|\beta|)}, \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n, \tag{2.15}$$

for all $\alpha, \beta \in \mathbb{N}^n_0$ and some constants $c_{\alpha,\beta} > 0$. The related pseudodifferential operator $p(x,\mathrm{D}) \in \Psi^\varkappa_{1,\gamma}$ is given, at least formally, by

$$p(x,\mathrm{D})f(x) = \int_{\mathbb{R}^n} e^{ix\xi} p(x,\xi)\mathcal{F}f(\xi)\,\mathrm{d}\xi. \tag{2.16}$$

As for the necessary background material we refer to [43,44,27]. Note that it is sufficient to study mapping properties of pseudodifferential operators for $\varkappa = 0$, since there is a unique relation between $\Psi^\varkappa_{1,\gamma}$ and $\Psi^0_{1,\gamma}$ given by

$$p(x,\mathrm{D}) = p^0(x,\mathrm{D})(\mathrm{id} - \Delta)^{\frac{\varkappa}{2}}, \quad \varkappa \in \mathbb{R}, \tag{2.17}$$

where Δ stands for the Laplacian in $\mathbb{R}^n$, $p(x,\mathrm{D}) \in \Psi^\varkappa_{1,\gamma}$ and $p^0(x,\mathrm{D}) \in \Psi^0_{1,\gamma}$. Moreover, we shall restrict ourselves to the situation $\gamma = 0$ in the sequel (avoiding, in particular, the *exotic case* $\gamma = 1$).

The following result can be found in [7, Theorem 2.8, Remark 3.4(c)].

Proposition 2.6. *Let $0 < p < \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$, $w \in \mathcal{A}_\infty$.*

(i) *The operator J_σ, given by $\mathcal{F} J_\sigma(x) = (1 + 4\pi^2|x|^2)^{-\sigma/2}$, $\sigma \in \mathbb{R}$, is an isomorphism of $A^s_{p,q}(\mathbb{R}^n, w)$ onto $A^{s+\sigma}_{p,q}(\mathbb{R}^n, w)$.*
(ii) *The operator $p(x, \mathrm{D}) \in \Psi^0_{1,0}$ is bounded on $A^s_{p,q}(\mathbb{R}^n, w)$.*

3. Embeddings

3.1. Continuity and compactness of embeddings

We collect some embedding results for weighted spaces of the above type that will be used later.

Proposition 3.1.

(i) *Let $0 < q \leqslant \infty$, $0 < p < \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty$. Then*

$$B^s_{p,\min(p,q)}\big(\mathbb{R}^n, w\big) \hookrightarrow F^s_{p,q}\big(\mathbb{R}^n, w\big) \hookrightarrow B^s_{p,\max(p,q)}\big(\mathbb{R}^n, w\big). \tag{3.1}$$

(ii) *Let $1 < p < \infty$, $-n < \alpha_1 < n(p-1)$, $-n < \beta_1$ and $\alpha_2, \beta_2 \in \mathbb{R}$, then*

$$B^0_{p,\min(p,2)}\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big) \hookrightarrow L_p\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big) \hookrightarrow B^0_{p,\max(p,2)}\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big).$$

If, in addition, $\beta_1 < n(p-1)$, then

$$L_p\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big) = F^0_{p,2}\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big).$$

Proof. The embeddings (i) are natural extensions from the unweighted case $w \equiv 1$, see [46, Proposition 2.3.2/2]. The result essentially coincides with [7, Theorem 2.6], see also [23, Proposition 1.8]. So we concentrate on the embeddings (ii).

Let φ be a smooth function such that $\operatorname{supp}\varphi \subset B(0, 2R)$ and $\varphi(x) = 1$ if $|x| \leqslant R$, where R is a positive sufficiently large number. Using the wavelet characterisation of weighted Besov spaces with Muckenhoupt weights, cf. [23], one can easily prove that

$$\big\|\varphi f \big| B^0_{p,q}\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big)\big\| \sim \big\|\varphi f \big| B^0_{p,q}\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\alpha})}\big)\big\|. \tag{3.2}$$

Similarly taking into account the wavelet characterisation of the weighted Besov spaces with admissible weights, cf. [26] one gets

$$\big\|(1-\varphi) f \big| B^0_{p,q}\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big)\big\| \sim \big\|(1-\varphi) f \big| B^0_{p,q}\big(\mathbb{R}^n, \big(1+|x|^2\big)^{\beta/2}\big)\big\|. \tag{3.3}$$

Consequently, by (3.2) we get

$$\begin{aligned} \big\|\varphi f \big| B^0_{p,\max(p,2)}\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big)\big\| &\leqslant C \big\|\varphi f \big| B^0_{p,\max(p,2)}\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\alpha})}\big)\big\| \\ &\leqslant C \big\|\varphi f \big| L_p\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\alpha})}\big)\big\| \\ &\leqslant C \big\|\varphi f \big| L_p\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big)\big\|, \end{aligned} \tag{3.4}$$

since $w_{(\boldsymbol{\alpha},\boldsymbol{\alpha})} \in \mathcal{A}_p$. Similarly by (3.3),

$$\begin{aligned} \big\|(1-\varphi) f \big| B^0_{p,\max(p,2)}\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big)\big\| &\leqslant C \big\|(1-\varphi) f \big| B^0_{p,\max(p,2)}\big(\mathbb{R}^n, \big(1+|x|^2\big)^{\frac{\beta}{2}}\big)\big\| \\ &\leqslant C \big\|(1-\varphi) f \big| L_p\big(\mathbb{R}^n, \big(1+|x|^2\big)^{\frac{\beta}{2}}\big)\big\| \\ &\leqslant C \big\|(1-\varphi) f \big| L_p\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big)\big\|, \end{aligned} \tag{3.5}$$

since $L_p(\mathbb{R}^n, (1+|x|^2)^{\beta/2}) = F^0_{p,2}(\mathbb{R}^n, (1+|x|^2)^{\beta/2})$, cf. [20]. Now (3.4) and (3.5) imply

$$\big\|f \big| B^0_{p,\max(p,2)}\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big)\big\| \leqslant C \big\|f \big| L_p\big(\mathbb{R}^n, w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big)\big\|. \tag{3.6}$$

The proof of the second embedding is similar. $\square$

There is a general criterion in [23] for the embedding

$$B^{s_1}_{p_1,q_1}\big(\mathbb{R}^n, w_1\big) \hookrightarrow B^{s_2}_{p_2,q_2}\big(\mathbb{R}^n, w_2\big), \quad w_1, w_2 \in \mathcal{A}_\infty,$$

but we formulate it only adapted to our later needs, i.e., with $p_1 \leqslant p_2$, $s = s_1 > 0$, $s_2 = 0$. The complete result together with its F-counterpart can be found in [23]. Let for $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$, $Q_{\nu,m}$ denote the n-dimensional cube with sides parallel to the axes of coordinates, centered at $2^{-\nu}m$ and with side length $2^{-\nu}$.

Proposition 3.2. *Let w_1 and w_2 be two $\mathcal{A}_\infty$ weights and let $s>0$, $0<p_1\leqslant p_2\leqslant\infty$, $0<q_1,q_2\leqslant\infty$. We put*

$$\frac{1}{q^*} := \left(\frac{1}{q_2}-\frac{1}{q_1}\right)_+. \tag{3.7}$$

(i) *There is a continuous embedding $B^s_{p_1,q_1}(\mathbb{R}^n,w_1)\hookrightarrow B^0_{p_2,q_2}(\mathbb{R}^n,w_2)$ if, and only if,*

$$\left\{2^{-\nu s}\sup_{m\in\mathbb{Z}^n}\frac{w_2(Q_{\nu,m})^{1/p_2}}{w_1(Q_{\nu,m})^{1/p_1}}\right\}_{\nu\in\mathbb{N}_0}\in\ell_{q^*}. \tag{3.8}$$

(ii) *The embedding $B^s_{p_1,q_1}(\mathbb{R}^n,w_1)\hookrightarrow B^0_{p_2,q_2}(\mathbb{R}^n,w_2)$ is compact if, and only if,* (3.8) *holds and, in addition,*

$$\lim_{|m|\to\infty}\frac{w_2(Q_{\nu,m})^{1/p_2}}{w_1(Q_{\nu,m})^{1/p_1}}=0\quad\text{for all } \nu\in\mathbb{N}_0, \tag{3.9}$$

and

$$\lim_{\nu\to\infty}2^{-\nu s}\sup_{m\in\mathbb{Z}^n}\frac{w_2(Q_{\nu,m})^{1/p_2}}{w_1(Q_{\nu,m})^{1/p_1}}=0\quad\text{if } q^*=\infty. \tag{3.10}$$

We postpone the explication of Proposition 3.2 in terms of $w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ given by (2.5) to the next section where these embeddings are studied in further detail.

3.2. Entropy numbers

Let X, Y be two quasi-Banach spaces and let $T:X\to Y$ be a bounded linear operator. The k-th (dyadic) entropy number of T, $k\in\mathbb{N}$, is defined as

$$e_k(T)=\inf\bigl\{\varepsilon>0:\ T(B_X)\text{ can be covered by }2^{k-1}\text{ balls of radius }\varepsilon\text{ in }Y\bigr\},$$

where B_X denotes the closed unit ball in X. Due to the well-known fact that

$$T:X\to Y\text{ is compact}\quad\text{if, and only if,}\quad\lim_{k\to\infty}e_k(T:X\to Y)=0,$$

the entropy numbers can be viewed as a quantification of the notion of compactness. Further properties like multiplicativity and additivity, as well as applications can be found in [12,15,16,35].

We recall our main result in [25] for the embedding

$$\mathrm{id}^A_{\log}:A^s_{p_1,q_1}\bigl(\mathbb{R}^n,w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\bigr)\hookrightarrow A^0_{p_2,q_2}\bigl(\mathbb{R}^n\bigr), \tag{3.11}$$

restricted again to the case $p_1\leqslant p_2$ and $s=s_1>s_2=0$.

Theorem 3.3. *Let the parameters satisfy*

$$s>0,\qquad 0<p_1\leqslant p_2\leqslant\infty,\qquad 0<q_1,q_2\leqslant\infty, \tag{3.12}$$

and denote

$$\delta=s-\frac{n}{p_1}+\frac{n}{p_2}. \tag{3.13}$$

Let the weight $w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\in\mathcal{A}_\infty$ be of type (2.5) *with* (2.6)*. We assume that*

$$\textit{either}\quad\min\left(\delta,\frac{\beta_1}{p_1}\right)>0,\quad\beta_2\in\mathbb{R}, \tag{3.14}$$

$$\textit{or}\quad\beta_1=0,\quad\beta_2>0, \tag{3.15}$$

and

$$\textit{either}\quad\delta>\frac{\max(\alpha_1,0)}{p_1},\quad\alpha_2\in\mathbb{R}, \tag{3.16}$$

$$\textit{or}\quad\delta=\frac{\alpha_1}{p_1}>0,\quad\frac{\alpha_2}{p_1}>\frac{s}{n}+\frac{\delta}{n}. \tag{3.17}$$

(i) *If* $\frac{\beta_1}{p_1} \neq \delta$ *and* $\beta_2 \in \mathbb{R}$, *then for* $k \in \mathbb{N}$,

$$e_k\big(\mathrm{id}^A_{\log}\big) \sim \begin{cases} k^{-\frac{\beta_1}{np_1}-\frac{1}{p_1}+\frac{1}{p_2}}(1+\log k)^{-\frac{\beta_2}{p_1}} & \text{if } \frac{\beta_1}{p_1} < \delta, \\ k^{-\frac{s}{n}} & \text{if } \frac{\beta_1}{p_1} > \delta. \end{cases}$$

(ii) *If* $\delta = \frac{\beta_1}{p_1}$, $\beta_2 \in \mathbb{R}$, *and* $\tau = \frac{s}{n} + \frac{1}{q_2} - \frac{1}{q_1} \neq \max(0, \frac{\beta_2}{p_1})$, *then for* $k \in \mathbb{N}$,

$$e_k\big(\mathrm{id}^B_{\log}\big) \sim \begin{cases} k^{-\frac{s}{n}} & \text{if } \frac{\beta_2}{p_1} > \max(\tau, 0), \\ k^{-\frac{s}{n}}(1+\log k)^{\tau-\frac{\beta_2}{p_1}} & \text{if } \frac{\beta_2}{p_1} \leqslant 0 < \tau, \text{ or } 0 < \frac{\beta_2}{p_1} < \tau \leqslant \frac{\beta_2}{p_1} + \frac{1}{q_2} - \frac{1}{q_1}, \\ k^{-\frac{s}{n}}(1+\log k)^{-\frac{\beta_2}{p_1}} & \text{if } \beta_2 \leqslant 0, \tau < 0. \end{cases} \tag{3.18}$$

(iii) *In case of* $\delta > 0 = \beta_1$, $\beta_2 > 0$, *then for all* $k \in \mathbb{N}$,

$$e_k\big(\mathrm{id}^A_{\log}\big) \sim \begin{cases} k^{-\frac{\beta_2}{p_1}} & \text{if } \frac{\beta_2}{p_1} \leqslant \frac{1}{p_1} - \frac{1}{p_2}, \\ k^{-\frac{1}{p_1}+\frac{1}{p_2}}(1+\log k)^{-\frac{\beta_2}{p_1}+\frac{1}{p_1}-\frac{1}{p_2}} & \text{if } \frac{\beta_2}{p_1} > \frac{1}{p_1} - \frac{1}{p_2}. \end{cases}$$

Remark 3.4. In [23] we studied entropy (and approximation) numbers of the corresponding situation when $\alpha_2 = \beta_2 = 0$. For later use we shall mention that

$$e_k\big(B^s_{p_1,q_1}(w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) \hookrightarrow B^0_{p_2,q_2}(w_{(\boldsymbol{\gamma},\boldsymbol{\eta})})\big) \sim e_k\big(B^s_{p_1,q_1}(w_{(\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}})}) \hookrightarrow B^0_{p_2,q_2}\big), \tag{3.19}$$

where

$$\frac{\widetilde{\alpha}_i}{p_1} = \frac{\alpha_i}{p_1} - \frac{\gamma_i}{p_2}, \qquad \frac{\widetilde{\beta}_i}{p_1} = \frac{\beta_i}{p_1} - \frac{\eta_i}{p_2}, \quad i = 1, 2, \tag{3.20}$$

which follows from the proofs in [23,25], essentially based on the isomorphisms

$$\begin{array}{ccc} B^s_{p_1,q_1}(\mathbb{R}^n, w_1) & \underset{A^{-1}}{\overset{A}{\rightleftarrows}} & B^s_{p_1,q_1}(\mathbb{R}^n, \widetilde{w}) \\ \Big\downarrow{\scriptstyle \mathrm{id}_{ww}} & & \Big\downarrow{\scriptstyle \mathrm{id}_B} \\ B^0_{p_2,q_2}(\mathbb{R}^n, w_2) & \underset{A}{\overset{A^{-1}}{\leftrightarrows}} & B^0_{p_2,q_2}(\mathbb{R}^n) \end{array} \tag{3.21}$$

see [23], with

$$\frac{w_2(Q_{\nu,m})^{1/p_2}}{w_1(Q_{\nu,m})^{1/p_1}} \sim \frac{|Q_{\nu,m}|^{1/p_2}}{\widetilde{w}(Q_{\nu,m})^{1/p_1}}.$$

4. Eigenvalue distribution of degenerate elliptic operators

4.1. Some preparation

Let X be a complex quasi-Banach space, $B : X \to X$ compact and linear with eigenvalue sequence $\{\mu_k(B)\}_{k\in\mathbb{N}}$, counted with respect to their algebraic multiplicity and ordered by decreasing modulus. If B has only finitely many distinct eigenvalues different from zero and k is the sum of their algebraic multiplicities, then we put $\mu_m(B) = 0$ for $m > k$. Then *Carl's inequality* [11,13], see also [12], gives

$$\left(\prod_{j=1}^{k} \big|\mu_j(B)\big|\right)^{\frac{1}{k}} \leqslant \inf_{m\in\mathbb{N}} 2^{\frac{m}{2k}} e_m(B) \quad \text{for all } k \in \mathbb{N}, \tag{4.1}$$

especially,

$$\big|\mu_k(B)\big| \leqslant \sqrt{2} e_k(B) \quad \text{for all } k \in \mathbb{N}. \tag{4.2}$$

Proposition 4.1. *Let b_1, b_2 be measurable functions and let $w_1, w_2 \in \mathcal{A}_\infty$ be Muckenhoupt weights. Let $1 < p < \infty$ and $\varkappa > 0$. We choose $1 < r_1, r_2 \leqslant \infty$ such that*

$$1 \leqslant r_1' < p < r_2 \leqslant \infty \quad \text{and} \quad \varkappa > n\left(\frac{1}{r_1} + \frac{1}{r_2}\right). \tag{4.3}$$

Moreover, let $w_1 \in \mathcal{A}_{q_1}$ and $w_2 \in \mathcal{A}_{q_2}$ with

$$\frac{1}{q_2} := \frac{1}{p} - \frac{1}{r_2} \quad \text{and} \quad \frac{1}{q_1} := \frac{1}{p} + \frac{1}{r_1}. \tag{4.4}$$

Let

$$\frac{1}{q} = \frac{1}{\min(q_2, 2)} - \frac{1}{\max(q_1, 2)} \geqslant 0. \tag{4.5}$$

Then the operator

$$B = b_2(\mathrm{id} - \Delta)^{-\varkappa/2} b_1 \tag{4.6}$$

is compact in $L_p(\mathbb{R}^n)$ if

$$b_1 w_1^{\frac{1}{q_1}} \in L_{r_1}(\mathbb{R}^n), \qquad b_2 w_2^{-\frac{1}{q_2}} \in L_{r_2}(\mathbb{R}^n), \tag{4.7}$$

and

$$\left\{2^{-\nu\varkappa} \sup_{m \in \mathbb{Z}^n} \frac{w_2(Q_{\nu,m})^{1/q_2}}{w_1(Q_{\nu,m})^{1/q_1}}\right\}_\nu \in \ell_q, \tag{4.8}$$

$$\lim_{|m| \to \infty} \frac{w_2(Q_{\nu,m})^{1/q_2}}{w_1(Q_{\nu,m})^{1/q_1}} = 0 \quad \text{for any } \nu \in \mathbb{N}_0, \tag{4.9}$$

and

$$\lim_{\nu \to \infty} 2^{-\nu\varkappa} \sup_{m \in \mathbb{Z}^n} \frac{w_2(Q_{\nu,m})^{1/q_2}}{w_1(Q_{\nu,m})^{1/q_1}} = 0 \quad \text{when } q = \infty. \tag{4.10}$$

Moreover,

$$\begin{aligned} \left|\mu_k(B)\right| \leqslant{}& c\left\|b_1 w_1^{\frac{1}{q_1}} \middle| L_{r_1}(\mathbb{R}^n)\right\| \left\|b_2 w_2^{-\frac{1}{q_2}} \middle| L_{r_2}(\mathbb{R}^n)\right\| \\ &\times e_k\left(B^{\varkappa}_{q_1,\max(q_1,2)}(\mathbb{R}^n, w_1) \hookrightarrow B^0_{q_2,\min(q_2,2)}(\mathbb{R}^n, w_2)\right), \quad k \in \mathbb{N}. \end{aligned} \tag{4.11}$$

Proof. We decompose B given by (4.6) as shown in Fig. 1, where in the usual $(\frac{1}{p}, s)$-diagram each point corresponds to spaces with the given smoothness and integrability (neglecting other parameters). Moreover, we now consider the functions b_i given by (4.7) as multiplication operators between spaces, $b_i : f \mapsto b_i f$, $i = 1, 2$ (in a slight abuse of notation). Then B can be understood as

$$B = b_2 \circ \mathrm{id}_w \circ (\mathrm{id} - \Delta)^{-\frac{\varkappa}{2}} \circ b_1,$$

with

$$b_1 : L_p \to B^0_{q_1,\max(q_1,2)}(w_1), \tag{4.12}$$

$$(\mathrm{id} - \Delta)^{-\frac{\varkappa}{2}} : B^0_{q_1,\max(q_1,2)}(w_1) \to B^{\varkappa}_{q_1,\max(q_1,2)}(w_1), \tag{4.13}$$

$$\mathrm{id}_w : B^{\varkappa}_{q_1,\max(q_1,2)}(w_1) \to B^0_{q_2,\min(q_2,2)}(w_2), \tag{4.14}$$

$$b_2 : B^0_{q_2,\min(q_2,2)}(w_2) \to L_p. \tag{4.15}$$

The boundedness of b_1 and b_2 in (4.12), (4.15) is due to Hölder's inequality in view of (4.7) together with the embeddings (3.1) and the identification $L_u(w) = F^0_{u,2}(w)$, justified by (2.14), when $w \in \mathcal{A}_u$. Here we apply $w_i \in \mathcal{A}_{q_i}$ and $1 < q_i < \infty$ by (4.3) and (4.4), $i = 1, 2$. The lift (4.13) is bounded by Proposition 2.6(i). Hence it remains to verify the compactness of id_w in (4.14). Moreover, the multiplicativity of entropy numbers leads in the same way to

$$e_k(B) \leqslant c\left\|b_1 w_1^{\frac{1}{q_1}} \middle| L_{r_1}\right\| \left\|b_2 w_2^{-\frac{1}{q_2}} \middle| L_{r_2}\right\| e_k(\mathrm{id}_w), \quad k \in \mathbb{N},$$

where c depends on the other parameters, but not on $k \in \mathbb{N}$. But this is covered by Proposition 3.2. $\square$

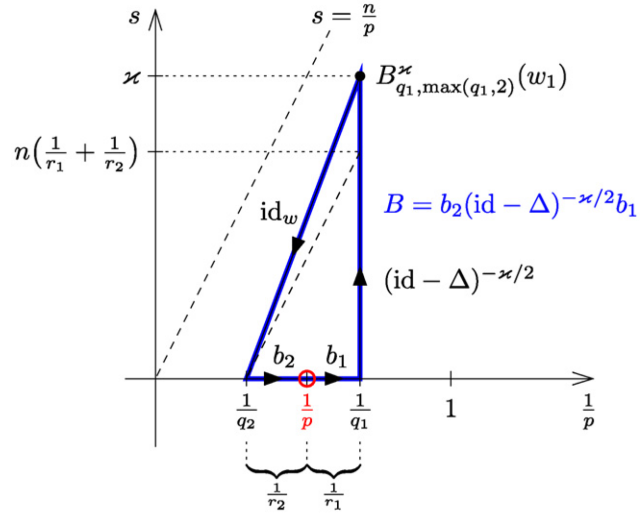

Fig. 1.

Remark 4.2. Note that our assumptions $w_i \in \mathcal{A}_{q_i}$, $i = 1, 2$, are only used to ensure the embeddings (3.1) with $s = 0$. Otherwise $w_i \in \mathcal{A}_\infty$, $i = 1, 2$, is sufficient (in addition to (4.7)–(4.10) which remain unchanged). Thus, in the particular case covered by Proposition 3.1(ii) for weights of type $w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, one may weaken the assumptions accordingly.

Remark 4.3. Let $v \in \mathcal{A}_\infty$ and assume that (4.7) can be replaced by

$$b_1 w_1^{\frac{1}{q_1}} v^{-\frac{1}{p}} \in L_{r_1}(\mathbb{R}^n) \quad \text{and} \quad b_2 w_2^{-\frac{1}{q_2}} v^{\frac{1}{p}} \in L_{r_2}(\mathbb{R}^n). \tag{4.16}$$

Then we immediately obtain a generalisation of the above proposition in the sense that B given by (4.6) acts compactly in $L_p(\mathbb{R}^n, v)$ with the counterpart of (4.11) given by

$$\begin{aligned}\left|\mu_k\big(B : L_p(\mathbb{R}^n, v) \to L_p(\mathbb{R}^n, v)\big)\right| &\leqslant c \left\|b_1 w_1^{\frac{1}{q_1}} v^{-\frac{1}{p}} \middle| L_{r_1}(\mathbb{R}^n)\right\| \left\|b_2 w_2^{-\frac{1}{q_2}} v^{\frac{1}{p}} \middle| L_{r_2}(\mathbb{R}^n)\right\| \\ &\quad \times e_k\big(B^{\varkappa}_{q_1,\max(q_1,2)}(\mathbb{R}^n, w_1) \hookrightarrow B^0_{q_2,\min(q_2,2)}(\mathbb{R}^n, w_2)\big), \quad k \in \mathbb{N},\end{aligned}$$

assuming all other assumptions in the proposition to be satisfied; hence (4.11) corresponds to $v \equiv 1$. This follows by Hölder's inequality in view of (4.12), (4.15) and the above argument.

4.2. The main result

Theorem 4.4. *Let $\varkappa > 0$, $1 < p < \infty$, and $1 < r_1, r_2 \leqslant \infty$ satisfy the assumptions* (4.3) *and* (4.4)*. Let $\alpha_i, \beta_i, \gamma_i, \eta_i \in \mathbb{R}$, $i = 1, 2$, with*

$$b_1 w_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \in L_{r_1}(\mathbb{R}^n) \quad \text{with } -\frac{n}{q_1} < \alpha_1 < \frac{n}{q_1'},\ \beta_1 > -\frac{n}{q_1}, \tag{4.17}$$

and

$$b_2 w_{(\boldsymbol{\gamma},\boldsymbol{\eta})} \in L_{r_2}(\mathbb{R}^n) \quad \text{with } -\frac{n}{q_2'} < \gamma_1 < \frac{n}{q_2},\ \eta_1 < \frac{n}{q_2}. \tag{4.18}$$

Moreover we assume that

$$\beta_1 + \eta_1 > 0 \quad \text{or} \quad \beta_1 + \eta_1 = 0 \quad \text{and} \quad \beta_2 + \eta_2 > 0, \tag{4.19}$$

and

$$\varkappa - n\left(\frac{1}{r_1} + \frac{1}{r_2}\right) > (\alpha_1 + \gamma_1)_+, \tag{4.20}$$

or

$$\varkappa - n\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = \alpha_1 + \gamma_1 > 0 \quad \text{and} \quad \alpha_2 + \gamma_2 > 2\frac{\varkappa}{n} - \frac{1}{r_1} - \frac{1}{r_2}. \tag{4.21}$$

Then the operator

$$B = b_2(\mathrm{id} - \Delta)^{-\varkappa/2} b_1 \tag{4.22}$$

is compact in $L_p(\mathbb{R}^n)$ *with*

$$\begin{aligned}
\bigl|\mu_k(B)\bigr| &\leqslant c \bigl\| b_1 w_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \big| L_{r_1}\bigl(\mathbb{R}^n\bigr)\bigr\| \bigl\| b_2 w_{(\boldsymbol{\gamma},\boldsymbol{\eta})} \big| L_{r_2}\bigl(\mathbb{R}^n\bigr)\bigr\| \\
&\quad \times \begin{cases}
k^{-\frac{\beta_1+\eta_1}{n} - \frac{1}{r_1} - \frac{1}{r_2}} (1+\log k)^{-\beta_2-\eta_2} & \text{if } 0 < \beta_1+\eta_1 < \varkappa - \frac{n}{r_1} - \frac{n}{r_2}, \\
k^{-\frac{\varkappa}{n}} & \text{if } \beta_1+\eta_1 > \varkappa - \frac{n}{r_1} - \frac{n}{r_2}, \\
& \quad \text{or } \beta_1+\eta_1 = \varkappa - \frac{n}{r_1} - \frac{n}{r_2}, \\
& \quad \text{and } \beta_2+\eta_2 > \tau, \\
k^{-\frac{\varkappa}{n}} (1+\log k)^{\tau-\beta_2-\eta_2} & \text{if } \beta_1+\eta_1 = \varkappa - \frac{n}{r_1} - \frac{n}{r_2}, \\
& \quad \text{and } \beta_2+\eta_2 \leqslant 0, \\
& \quad \text{or } 0 < \beta_2+\eta_2 < \tau \leqslant \beta_2+\eta_2 + \frac{1}{q}, \\
k^{-\beta_2-\eta_2} & \text{if } \beta_1+\eta_1 = 0, \\
& \quad \text{and } 0 < \beta_2+\eta_2 \leqslant \frac{1}{r_1} + \frac{1}{r_2}, \\
k^{-\frac{1}{r_1} - \frac{1}{r_2}} (1+\log k)^{-\beta_2-\eta_2+\frac{1}{r_1}+\frac{1}{r_2}} & \text{if } \beta_1+\eta_1 = 0, \\
& \quad \text{and } \beta_2+\eta_2 > \frac{1}{r_1} + \frac{1}{r_2},
\end{cases}
\end{aligned} \tag{4.23}$$

where $\tau = \frac{\varkappa}{n} + \frac{1}{q}$ *and* q *is given by* (4.5).

Proof. We apply Proposition 4.1 with

$$w_1 = (w_{(\boldsymbol{\alpha},\boldsymbol{\beta})})^{q_1} = w_{(\boldsymbol{\alpha q_1},\boldsymbol{\beta q_1})} \quad \text{and} \quad w_2 = (w_{(\boldsymbol{\gamma},\boldsymbol{\eta})})^{-q_2} = w_{(-\boldsymbol{\gamma q_2},-\boldsymbol{\eta q_2})}.$$

Moreover, in view of Remark 3.4 we obtain all other assertions from Theorem 3.3 applied to

$$\mathrm{id} : B^{\varkappa}_{q_1,\max(q_1,2)}(w_{(\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}})}) \hookrightarrow B^0_{q_2,\min(q_2,2)}, \tag{4.24}$$

where

$$\frac{\widetilde{\alpha}_i}{q_1} = \frac{\alpha_i q_1}{q_1} - \frac{(-\gamma_i q_2)}{q_2} = \alpha_i + \gamma_i, \quad i = 1, 2,$$
$$\frac{\widetilde{\beta}_i}{q_1} = \frac{\beta_i q_1}{q_1} - \frac{(-\eta_i q_2)}{q_2} = \beta_i + \eta_i, \quad i = 1, 2,$$

recall (3.19) and (3.20). Note that (4.17) and (4.18) imply that we can apply Proposition 3.1(ii) to $w_1 = w_{(\boldsymbol{\alpha q_1},\boldsymbol{\beta q_1})}$ and $w_2 = w_{(-\boldsymbol{\gamma q_2},-\boldsymbol{\eta q_2})}$ which in view of Remark 4.2 is sufficient to apply Proposition 4.1 in this special case. Moreover, δ given by (3.13) reads in case of (4.24) as $\delta = \varkappa - \frac{n}{q_1} + \frac{n}{q_2} = \varkappa - n(\frac{1}{r_1} + \frac{1}{r_2})$, recall (4.4). $\square$

For later use we specify Theorem 4.4 for the case $b_1 = b_2 = b$, that is,

$$B = b(\mathrm{id} - \Delta)^{-\varkappa/2} b. \tag{4.25}$$

Moreover, since we are finally interested in the case $p = 2$ only, we shall also restrict ourselves to this case which implies $q = \infty$ in (4.5); see also Fig. 1.

Corollary 4.5. *Let* $2 < r \leqslant \infty$, $\frac{\varkappa}{2} > \frac{n}{r}$. *Let* $w_{(\boldsymbol{\alpha},\boldsymbol{\beta})} b \in L_r(\mathbb{R}^n)$ *with* $\alpha_2, \beta_2 \in \mathbb{R}$ *and*

$$-n\left(\frac{1}{2} + \frac{1}{r}\right) < \min(\alpha_1, \beta_1) \leqslant \max(\alpha_1, \beta_1) < n\left(\frac{1}{2} - \frac{1}{r}\right). \tag{4.26}$$

Moreover we assume that

$$\beta_1 > 0 \quad \text{or} \quad \beta_1 = 0 \quad \text{and} \quad \beta_2 > 0, \tag{4.27}$$

and

$$\frac{\varkappa}{2}-\frac{n}{r}>(\alpha_1)_+, \quad \text{or} \quad \frac{\varkappa}{2}-\frac{n}{r}=\alpha_1>0 \quad \text{and} \quad \alpha_2>\frac{\varkappa}{n}-\frac{1}{r}. \tag{4.28}$$

Then the operator

$$B=b(\mathrm{id}-\Delta)^{-\varkappa/2}b \tag{4.29}$$

is compact in $L_2(\mathbb{R}^n)$ *with*

$$\left|\mu_k(B)\right| \leqslant c\left\|w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}b\middle|L_r\left(\mathbb{R}^n\right)\right\|^2 \begin{cases} k^{-\frac{2}{n}\beta_1-\frac{2}{r}}(1+\log k)^{-2\beta_2} & \text{if } 0<\beta_1<\frac{\varkappa}{2}-\frac{n}{r}, \\ k^{-\frac{\varkappa}{n}} & \text{if } \beta_1>\frac{\varkappa}{2}-\frac{n}{r}, \\ & \quad \text{or } \beta_1=\frac{\varkappa}{2}-\frac{n}{r} \text{ and } 2\beta_2>\frac{\varkappa}{n}, \\ k^{-\frac{\varkappa}{n}}(1+\log k)^{\frac{\varkappa}{n}-2\beta_2} & \text{if } \beta_1=\frac{\varkappa}{2}-\frac{n}{r} \text{ and } \beta_2\leqslant 0, \\ k^{-2\beta_2} & \text{if } \beta_1=0 \text{ and } 0<\beta_2\leqslant\frac{1}{r}, \\ k^{-\frac{2}{r}}(1+\log k)^{-2\beta_2+\frac{2}{r}} & \text{if } \beta_1=0 \text{ and } \beta_2>\frac{1}{r}. \end{cases} \tag{4.30}$$

Remark 4.6. Since we have $q=\infty$ now, the case $\beta_2+\eta_2<\tau\leqslant\beta_2+\eta_2+\frac{1}{q}$ in (4.23) disappears.

Remark 4.7. Let $p(x,\mathrm{D})\in\Psi_{1,0}^{-\varkappa}$ be a pseudodifferential operator of order $-\varkappa<0$ belonging to the Hörmander class $\Psi_{1,0}^{-\varkappa}$. It follows from Proposition 2.6 that Proposition 4.1, Theorem 4.4 and Corollary 4.5 hold for the operator

$$B=b_2\,p(x,\mathrm{D})b_1$$

instead of $b_2(\mathrm{id}-\Delta)^{-\varkappa/2}b_1$.

5. The negative spectrum

The interest in studying the 'negative' spectrum (bound states) comes from quantum mechanics, generalising the classical hydrogen operator,

$$H=-\Delta-\frac{c}{|x|}, \quad c>0, \tag{5.1}$$

in $L_2(\mathbb{R}^3)$. Thus 'potentials' $V(x)$ with $V(x)\sim|x|^{-a}$, $a>0$, are of peculiar interest, or, more general, 'potentials' which have local singularities and some decay properties at infinity. We first describe the general setting briefly, before we apply our results from the previous section to this situation. Afterwards we can cover some more limiting cases using splitting techniques.

5.1. Birman–Schwinger principle

We adapt the Birman–Schwinger principle as described in [38,40] to our concrete situation. Let A be a self-adjoint positive-definite operator and let B be a symmetric relatively compact operator in the Hilbert space L_2. Let σ_{p} be the point spectrum and σ_{e} be the essential spectrum. Then the eigenvalues $\{\mu_k\}_k$ of BA^{-1} are real, and $(BA^{-1})^*=A^{-1}B$ is the adjoint operator after extension by continuity from $\mathrm{dom}(B)$ to L_2. Furthermore, the operator $A+B$ with $\mathrm{dom}(A+B)=\mathrm{dom}(A)$, is self-adjoint, with $\sigma_{\mathrm{e}}(A+B)=\sigma_{\mathrm{e}}(A)$, and

$$\begin{aligned}\#\left\{\sigma_{\mathrm{p}}(A+B)\cap(-\infty,0]\right\}&=\#\left\{\sigma(A+B)\cap(-\infty,0]\right\}\\ &=\#\left\{k\in\mathbb{N}\colon\ \mu_k\left(BA^{-1}\right)\leqslant-1\right\}<\infty.\end{aligned} \tag{5.2}$$

This is usually called the *Birman–Schwinger principle*. It goes back to [1,39], proofs may be found in [40, Chapter 7] and [38, Chapter 8, §5]. A short description has also been given in [16, Section 5.2.1, p. 186]. Our formulation is different and adapted to our later needs. Of course, eigenvalues are counted according to their multiplicities. Using (4.2) with $T=BA^{-1}$ one obtains by (5.2) that

$$\#\left\{\sigma_{\mathrm{p}}(A+B)\cap(-\infty,0]\right\}\leqslant\#\left\{k\in\mathbb{N}\colon\ \sqrt{2}e_k\left(BA^{-1}\right)\geqslant 1\right\}. \tag{5.3}$$

This entropy version of the Birman–Schwinger principle appeared first in [20, Theorem 2.4], cf. also [16, Corollary, p. 186].

We shall concentrate on the special case when $B = -V$ is a multiplication operator where (in a slight abuse of notation) V is a nonnegative measurable function, finite a.e., typically belonging to some space $L_r(w)$, $w \in \mathcal{A}_\infty$. Let L_2 be always the basic space in the sequel.

We turn to study the behaviour of the negative spectrum of the self-adjoint unbounded operator

$$H_\gamma = A - \gamma V \quad \text{as } \gamma \to \infty, \tag{5.4}$$

and can thus conclude that

$$\#\{\sigma_{\mathrm{p}}(H_\gamma) \cap (-\infty, 0]\} \leqslant \#\{k \in \mathbb{N}\colon \sqrt{2} e_k(VA^{-1}) \geqslant \gamma^{-1}\}, \tag{5.5}$$

or, more adapted to our formulations in Section 4,

$$\#\{\sigma_{\mathrm{p}}(H_\gamma) \cap (-\infty, 0]\} \leqslant \#\{k \in \mathbb{N}\colon \sqrt{2} e_k(V^{\frac{1}{2}} A^{-1} V^{\frac{1}{2}}) \geqslant \gamma^{-1}\}. \tag{5.6}$$

5.2. The basic result

Now we present our main result concerning the negative spectrum in case of positive, elliptic, pseudodifferential operators of order $\varkappa > 0$, $A \in \Psi^{\varkappa}_{1,0}$, self-adjoint in $L_2(\mathbb{R}^n)$, that is, for

$$H_\gamma = A - \gamma V. \tag{5.7}$$

This covers, in particular, the setting $H_\gamma = (\mathrm{id} - \Delta)^{\frac{\varkappa}{2}} - \gamma V$. We apply our results from Section 4 to (5.6). We shall use Theorem 4.4 with $p = 2$ and Corollary 4.5 with $b = V^{\frac{1}{2}}$. Let $1 < r \leqslant \infty$ and $\varkappa > 0$. We define the following function for $k \in \mathbb{N}$,

$$\eta_{\boldsymbol{\beta},\varkappa,r}(k) = \begin{cases} k^{-\frac{\beta_1}{n} - \frac{1}{r}} (1 + \log k)^{-\beta_2} & \text{if } 0 < \beta_1 < \varkappa - \frac{n}{r}, \\ k^{-\frac{\varkappa}{n}} & \text{if } \beta_1 > \varkappa - \frac{n}{r}, \text{ or } \beta_1 = \varkappa - \frac{n}{r} \text{ and } \beta_2 > \frac{\varkappa}{n}, \\ k^{-\frac{\varkappa}{n}} (1 + \log k)^{\frac{\varkappa}{n} - \beta_2} & \text{if } \beta_1 = \varkappa - \frac{n}{r} \text{ and } \beta_2 \leqslant 0, \\ k^{-\beta_2} & \text{if } \beta_1 = 0 \text{ and } 0 < \beta_2 \leqslant \frac{1}{r}, \\ k^{-\frac{1}{r}} (1 + \log k)^{-\beta_2 + \frac{1}{r}} & \text{if } \beta_1 = 0 \text{ and } \beta_2 > \frac{1}{r}. \end{cases} \tag{5.8}$$

The function $\eta_{\boldsymbol{\beta},\varkappa,r}$ is decreasing, at least for sufficiently large k. Let $\eta^{-1}_{\boldsymbol{\beta},\varkappa,r}$ denote the function inverse to $\eta_{\boldsymbol{\beta},\varkappa,r}$.

Theorem 5.1. *Let A be a positive, elliptic, self-adjoint operator in $L_2(\mathbb{R}^n)$ of order $\varkappa > 0$ and H_γ be given by (5.7). Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and $1 < r_\alpha, r_\beta \leqslant \infty$. Assume that*

$$-n\left(1 + \frac{1}{r_\alpha}\right) < \alpha_1 < n\left(1 - \frac{1}{r_\alpha}\right), \tag{5.9}$$

and

$$0 < \beta_1 < n\left(1 - \frac{1}{r_\beta}\right), \quad \textit{or} \quad \beta_1 = 0 \quad \textit{and} \quad \beta_2 > 0. \tag{5.10}$$

Moreover we assume that

$$\varkappa > \max\left(\frac{n}{r_\alpha} + (\alpha_1)_+, \frac{n}{r_\beta}\right), \tag{5.11}$$

or

$$\varkappa > \frac{n}{r_\beta}, \qquad \varkappa - \frac{n}{r_\alpha} = \alpha_1 > 0 \quad \textit{and} \quad \alpha_2 > 2\frac{\varkappa}{n} - \frac{1}{r_\alpha}. \tag{5.12}$$

Let V be a nonnegative measurable function, finite a.e. and let $V_1 = \chi_{B(0,1)} V$ and $V_2 = V - V_1$. We assume that

(i) $\int_{\mathbb{R}^n} V_1(x)^{r_\alpha} |x|^{r_\alpha \alpha_1} (1 - \log|x|)^{r_\alpha \alpha_2}\, \mathrm{d}x = 1$,
(ii) $w_{(\mathbf{0},\boldsymbol{\beta})} V_2 \in L_{r_\beta}(\mathbb{R}^n)$.

Then there exists a positive constant $C > 0$ independent of γ, $\boldsymbol{\beta}$ and V such that

$$\#\{\sigma_{\mathrm{p}}(H_\gamma) \cap (-\infty, 0]\} \leqslant C \eta^{-1}_{\boldsymbol{\beta},\varkappa,r_\beta}(\gamma^{-1}), \quad \gamma \to \infty. \tag{5.13}$$

Proof. Using (5.6) we can reduce the proof to Theorem 4.4 and Corollary 4.5 with $p=2$. We split the operator $B=bA^{-1}b$, with $b=V^{\frac{1}{2}}$ into four parts,

$$B=B^{1,1}+B^{1,2}+B^{2,1}+B^{2,2}, \qquad B^{i,j}=b_iA^{-1}b_j, \quad i,j=1,2,$$

where $b_1=V_1^{\frac{1}{2}}$ and $b_2=V_2^{\frac{1}{2}}$.

To estimate the entropy numbers of the operator $B^{2,2}$ we can deal as in the case of Corollary 4.5 with $r=2r_\beta$ and the weight $w_{(\mathbf{0},\boldsymbol{\beta}/\mathbf{2})}$. The assumptions (5.9)–(5.11) imply (4.26)–(4.28) so we get

$$e_k\big(B^{2,2}\big)\leqslant C\eta_{\boldsymbol{\beta},\varkappa,r_\beta}(k). \tag{5.14}$$

Now we estimate the entropy numbers of the operator $B^{1,1}$. This can be done in a way similar to the proof of Theorem 4.4 with $r_1=r_2=2r_\alpha$. The support of V_1 is contained in $B(0,1)$, therefore by (i),

$$V_1^{\frac{1}{2}}w_{(\boldsymbol{\alpha}/\mathbf{2},\widetilde{\boldsymbol{\beta}})}\in L_{2r_\alpha}\big(\mathbb{R}^n\big) \tag{5.15}$$

and

$$V_1^{\frac{1}{2}}w_{(\boldsymbol{\alpha}/\mathbf{2},\mathbf{0})}\in L_{2r_\alpha}\big(\mathbb{R}^n\big) \tag{5.16}$$

with $\widetilde{\boldsymbol{\beta}}=(\widetilde{\beta}_1,0)$, $\widetilde{\beta}_1>\varkappa-\frac{n}{r_\alpha}$. Now with (5.15) corresponding to (4.17), and (5.16) to (4.18), assumptions (4.19)–(4.21) are consequences from (5.9)–(5.12). Hence we obtain,

$$e_k\big(B^{1,1}\big)\leqslant Ck^{-\varkappa/n}, \tag{5.17}$$

since $\widetilde{\beta}_1>\varkappa-\frac{n}{r_1}-\frac{n}{r_2}=\varkappa-\frac{n}{r_\alpha}$.

It remains to estimate the entropy numbers of the operators with mixed potentials $B^{1,2}$ and $B^{2,1}$. We give the proof for $B^{2,1}$, the argument for $B^{1,2}$ is similar. Once more we follow the ideas of the proof of Theorem 4.4, now taking $r_1=2r_\alpha$ and $r_2=2r_\beta$. By the localisation of the supports of V_1 and V_2 we have

$$V_1^{\frac{1}{2}}w_{(\boldsymbol{\alpha}/\mathbf{2},\widetilde{\boldsymbol{\beta}})}\in L_{2r_\alpha}\big(\mathbb{R}^n\big) \quad\text{and}\quad V_2^{\frac{1}{2}}w_{(\mathbf{0},\boldsymbol{\beta}/\mathbf{2})}\in L_{2r_\beta}\big(\mathbb{R}^n\big),$$

where $\widetilde{\boldsymbol{\beta}}=(\widetilde{\beta}_1,0)$ is chosen such that $\frac{\beta_1}{2}+\widetilde{\beta}_1>\varkappa-\frac{n}{2}(\frac{1}{r_\alpha}+\frac{1}{r_\beta})$. Now

$$\varkappa-n\bigg(\frac{1}{r_1}+\frac{1}{r_2}\bigg)=\frac{1}{2}\bigg(\varkappa-\frac{n}{r_\alpha}\bigg)+\frac{1}{2}\bigg(\varkappa-\frac{n}{r_\beta}\bigg)>\frac{(\alpha_1)_+}{2}$$

regardless whether (5.11) or (5.12) holds. Similarly one can check the assumptions (4.17)–(4.19). Thus we arrive at

$$e_k\big(B^{2,1}\big)\leqslant Ck^{-\varkappa/n}. \tag{5.18}$$

Changing the role of r_1 and r_2 we can prove the estimates for $B^{1,2}$. At the end the inequalities (5.14), (5.17) and (5.18) imply

$$e_k(B)\leqslant C\eta_{\boldsymbol{\beta},\varkappa,r_\beta}(k). \tag{5.19}$$

The rest follows by (5.19), (5.6) and (5.8). $\square$

Remark 5.2. In the assumptions of the last theorem we formulated different conditions (i) and (ii) for the potential at infinity and near the origin. It was already known before that such a splitting can be useful to deal with the negative spectrum, cf. [21]. Here we used this technique from the very beginning since the weights we deal with already possess this structure.

For power type weights $w_{\alpha,\beta}$ we have the following corollary that follows immediately from Theorem 5.1.

Corollary 5.3. *Let $\varkappa>0$, $w_{\alpha,\beta}V\in L_r(\mathbb{R}^n)$, and H_γ be given by* (5.7)*. Assume $1<r\leqslant\infty$, and*

$$-n\bigg(1+\frac{1}{r}\bigg)<\alpha<n\bigg(1-\frac{1}{r}\bigg), \qquad 0<\beta<n\bigg(1-\frac{1}{r}\bigg), \qquad \varkappa>\frac{n}{r}+\alpha_+. \tag{5.20}$$

Let $\varkappa\neq\beta+\frac{n}{r}$. Then

$$\#\big\{\sigma_{\mathrm{p}}(H_\gamma)\cap(-\infty,0]\big\}\leqslant c\big(\gamma\,\big\|V\big|L_r(w_{\alpha,\beta})\big\|^2\big)^{\frac{n}{\min(\varkappa,\beta+n/r)}}, \quad \gamma\to\infty. \tag{5.21}$$

Proof. We apply Theorem 5.1 with $r_\alpha = r_\beta = r$, $\alpha_2 = \beta_2 = 0$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, and use the first two lines of (5.8). $\square$

We would like to emphasise that Theorem 5.1 covers also some limiting cases.

Corollary 5.4. *Let A be a positive, elliptic self-adjoint operator in $L_2(\mathbb{R}^n)$ of order $\varkappa$, $0 < \varkappa < n$. Let V be a nonnegative measurable function, finite a.e., and assume $V w_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \in L_\infty(\mathbb{R}^n)$.*

(i) *Let $\alpha_1 = \varkappa$, $\alpha_2 > 2\frac{\varkappa}{n}$, $0 < \beta_1 = \beta < n$ and $\beta_2 = 0$. Then there exists a positive constant $C > 0$ independent of γ, $\boldsymbol{\beta}$ and V such that*

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \leqslant C \begin{cases} \gamma^{\frac{n}{\varkappa}} & \text{if } \varkappa < \beta, \\ \gamma^{\frac{n}{\beta}} & \text{if } \varkappa > \beta, \\ \gamma^{\frac{n}{\varkappa}} \log \gamma & \text{if } \varkappa = \beta, \end{cases} \quad \gamma \to \infty.$$

(ii) *Assume $\max(0, \alpha_1) < \varkappa$ and $\beta_1 = \varkappa$. Then there exists a positive constant $C > 0$ independent of γ, $\boldsymbol{\beta}$ and V such that*

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \leqslant C \begin{cases} \gamma^{\frac{n}{\varkappa}} & \text{if } \frac{\varkappa}{n} < \beta_2, \\ \gamma^{\frac{n}{\varkappa}} (\log \gamma)^{1-\beta_2 \frac{n}{\varkappa}} & \text{if } 0 \geqslant \beta_2, \end{cases} \quad \gamma \to \infty.$$

Proof. We apply Theorem 5.1 with $r_\alpha = r_\beta = \infty$ and use (5.8) according to the different situations. $\square$

5.3. Examples

We begin with the classical example of power type weights.

Example 5.5. Let $V(x) = |x|^{-\mu}$, and

$$H_\gamma = A - \gamma |x|^{-\mu}, \quad 0 < \mu < n, \ \varkappa > \mu.$$

Then

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \leqslant C \gamma^{\frac{n}{\mu}}, \quad \gamma \to \infty.$$

This is covered by Corollary 5.3 with $r = \infty$ and $\alpha = \beta = \mu$.

One can also consider potentials of logarithmic type.

Example 5.6. Let for $\alpha > 0$ and $\varkappa > 0$ the potential be

$$V(x) = \begin{cases} (1 - \log |x|)^\alpha & \text{if } |x| < 1, \\ (1 + \log |x|)^{-\alpha} & \text{if } |x| \geqslant 1, \end{cases}$$

and

$$H_\gamma = A - \gamma V(x).$$

Then

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \leqslant C \exp\left(\gamma^{\frac{1}{\alpha}}\right), \quad \gamma \to \infty.$$

Here we apply Theorem 5.1 with $r_\alpha = r_\beta = \infty$, $\alpha_1 = \beta_1 = 0$, $\alpha_2 = -\alpha$, $\beta_2 = \alpha$, such that (5.13) leads to

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \leqslant C \eta^{-1}_{\boldsymbol{\beta},\varkappa,\infty}\left(\gamma^{-1}\right), \quad \gamma \to \infty.$$

Since $\eta_{\boldsymbol{\beta},\varkappa,\infty}(k) \sim (1 + \log k)^{-\alpha}$ by the last line of (5.8), this completes the argument.

Combining both limiting situations we can deal with the following example.

Example 5.7. Let $0 < \varkappa < n$, $\varepsilon > 0$, and

$$V(x) = \begin{cases} |x|^{-\varkappa} (1 - \log |x|)^{-2\frac{\varkappa}{n} - \varepsilon} & \text{if } |x| < 1, \\ |x|^{-\varkappa} (1 + \log |x|)^{-\beta} & \text{if } |x| \geqslant 1. \end{cases}$$

Then for $\gamma \to \infty$,

$$\#\{\sigma_{\mathrm{p}}(H_\gamma) \cap (-\infty, 0]\} \leqslant C \begin{cases} \gamma^{\frac{n}{\varkappa}} & \text{if } \beta > \frac{\varkappa}{n}, \\ \gamma^{\frac{n}{\varkappa}} (\log \gamma)^{1+\frac{n}{\varkappa}(-\beta)_+} & \text{if } \beta < \frac{\varkappa}{n}. \end{cases}$$

This is a consequence of Theorem 5.1 with $r_\alpha = r_\beta = \infty$, $\alpha_1 = \beta_1 = \varkappa$, $\alpha_2 = 2\frac{\varkappa}{n} + \varepsilon$, and $\beta_2 = \beta$. Hence (5.13) leads to

$$\#\{\sigma_{\mathrm{p}}(H_\gamma) \cap (-\infty, 0]\} \leqslant C \eta^{-1}_{\boldsymbol{\beta},\varkappa,\infty}(\gamma^{-1}), \quad \gamma \to \infty.$$

In particular, if $\beta = 0$, then $\#\{\sigma_{\mathrm{p}}(H_\gamma) \cap (-\infty, 0]\} \leqslant C\gamma^{\frac{n}{\varkappa}} \log \gamma$. This answers a question put in [21]. Also for other values of β our estimates are more precise than the estimates given in [21, Theorem 5.3], since there the authors had to use polynomial weights with power $\alpha \neq \varkappa$.

At the end we give an example of a potential in the limiting case $\varkappa - \frac{n}{r_\alpha} = \alpha_1 > 0$ which is not covered by the choice $r_\alpha = \infty$ or by the unweighted setting $\alpha_1 = \alpha_2 = 0$ studied in [21].

Example 5.8. Let $0 < \varkappa < n$. We choose r positive such that $0 < \frac{n}{r} < \varkappa < n$ and a function $W \in L_r(\mathbb{R}^n)$ such that $\operatorname{supp} W \subset \{x \in \mathbb{R}^n\colon \frac{1}{2} \leqslant |x| \leqslant 1\}$. Let $2\frac{\varkappa}{n} > \sigma > 2\frac{\varkappa}{n} - \frac{1}{r}$. We put

$$V_1(x) = \sum_{j=0}^{\infty} 2^{j(\varkappa - \frac{n}{r})} (1+j)^{-\sigma} 2^{(j-1)n/r} W(2^{j-1}x).$$

For example, when $W(x) \equiv 1$, then

$$V_1(x) \sim j^{-\sigma} 2^{j\varkappa} \quad \text{for } 2^{-j} < |x| < 2^{-j+1},\ j = 1, 2, \ldots.$$

Consequently, for $\alpha_1 = \varkappa - \frac{n}{r} > 0$ and α_2 such that $\sigma > \alpha_2 > \frac{\varkappa}{n} - \frac{1}{r} > 0$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} V_1(x)^r |x|^{r\alpha_1} (1 - \log|x|)^{r\alpha_2} \, \mathrm{d}x &\leqslant C \sum_{j=1}^{\infty} 2^{-jr(\alpha_1 - \varkappa + \frac{n}{r})} j^{r(\alpha_2 - \sigma)} \int_{\mathbb{R}^n} 2^{(j-1)n} W^r(2^{(j-1)}x) \, \mathrm{d}x \\ &\sim C \| W | L_r(\mathbb{R}^n) \|^r \sum_{j=1}^{\infty} j^{r(\alpha_2 - \sigma)} < \infty, \end{aligned}$$

whereas V_1 does not belong to any (unweighted) space $L_u(\mathbb{R}^n)$ with $\varkappa > \frac{n}{u}$ (as required in [21]), in particular with $u = r$. Now let V be a potential such that $V|_{B(0,1)} = V_1$ and $V_2 = V - V_1$ satisfies the condition (ii) of Theorem 5.1. Then for the operator H_γ given by (5.7) the estimate (5.13) holds.

On the other hand, the assumption $r_\beta < \infty$ (also not covered by [21]) may be useful if V has singularities outside the origin. For example, with $0 < \frac{n}{r} < \varkappa < n$ and $W \in L_r(\mathbb{R}^n)$ as above, we can take

$$V_2(x) = \sum_{j=1}^{\infty} 2^{-j(\varkappa - \frac{n}{r})} (1+j)^{-\widetilde{\sigma}} 2^{-jn/r} W(2^{-j}x), \quad \text{where } \widetilde{\sigma} > \frac{\varkappa}{n} + \frac{1}{r}.$$

If $W \notin L_\infty(\mathbb{R}^n)$, then $w_{(\mathbf{0},\boldsymbol{\beta})} V_2 \notin L_\infty(\mathbb{R}^n)$ for any $\boldsymbol{\beta}$. But taking $r_\beta = r$, $\beta_1 = \varkappa - \frac{n}{r}$ and $\widetilde{\sigma} - \frac{1}{r} > \beta_2 > \frac{\varkappa}{n}$ we obtain by (5.13) the estimate

$$\#\{\sigma_{\mathrm{p}}(H_\gamma) \cap (-\infty, 0]\} \leqslant C\gamma^{\frac{n}{\varkappa}}, \quad \gamma \to \infty.$$

Remark 5.9. So far we adopted the traditional point of view asking for the behaviour of

$$\#\{\sigma_{\mathrm{p}}(H_\gamma) \cap (-\infty, 0]\} \quad \text{as } \gamma \to \infty, \tag{5.22}$$

where H_γ is given by (5.4). Looking at the classical background briefly described at the beginning of Section 5, then it seems to be even more natural to study

$$\#\{\sigma_{\mathrm{p}}(H) \cap (-\infty, -\varepsilon]\} \quad \text{as } \varepsilon \downarrow 0. \tag{5.23}$$

To be as close as possible to the hydrogen operator H in (5.1) we restrict ourselves to an example. Let

$$a(\mathrm{D}) = (-1)^m \sum_{|\alpha|=m} a_\alpha \mathrm{D}^{2\alpha} \tag{5.24}$$

with $a_\alpha \in \mathbb{R}$ and $\sum_{|\alpha|=m} a_\alpha \xi^{2\alpha} \geqslant c|\xi|^{2m}$, $\xi \in \mathbb{R}^n$; that is, an elliptic differential operator of order $2m$ with constant coefficients. Then $\sigma(a(\mathrm{D})) = \sigma_{\mathrm{e}}(a(\mathrm{D})) = [0, \infty)$, cf. [38] or [15, IX, 6,8]. In case of

$$H = a(\mathrm{D}) - |x|^{-\mu} \quad \text{with } 0 < \mu < \min(n, 2m), \tag{5.25}$$

we can apply Example 5.5 to $\widetilde{H}_\gamma = a(\mathrm{D}) - \gamma |x|^{-\mu}$ with $\varkappa = 2m$, and thus obtain for $H_\gamma = \mathrm{id} + \widetilde{H}_\gamma$ that

$$\#\bigl\{\sigma_{\mathrm{p}}(H_\gamma) \cap (-\infty, -1]\bigr\} \leqslant c\gamma^{\frac{n}{\mu}}. \tag{5.26}$$

On the other hand, due to the homogeneity properties of $a(\mathrm{D})$ and the potential $|x|^{-\mu}$ we observed in [21] that any eigenvalue λ of H_γ with $\lambda \leqslant -1$ corresponds to an eigenvalue $\lambda\gamma^{-\frac{2m}{2m-\mu}} =: \lambda\varepsilon$ of H, where one can choose $\gamma = \varepsilon^{-\frac{2m-\mu}{2m}}$ for given $\varepsilon > 0$. Thus we arrive at

$$\#\bigl\{\sigma_{\mathrm{p}}(H) \cap (-\infty, -\varepsilon]\bigr\} \leqslant c\varepsilon^{-n(\frac{1}{\mu} - \frac{1}{2m})}, \quad \varepsilon \downarrow 0, \tag{5.27}$$

as in [21, Theorem 5.4]. In particular, for the hydrogen atom (5.1) we have $n = 3$, $\mu = 1$ and $2m = 2$, such that (5.27) reads as

$$\#\bigl\{\sigma_{\mathrm{p}}(H) \cap (-\infty, -\varepsilon]\bigr\} \leqslant c\varepsilon^{-\frac{3}{2}}, \quad \varepsilon \downarrow 0. \tag{5.28}$$

This is well known, since the number of eigenvalues less than or equal to $-\varepsilon = -\frac{1}{4}N^{-2}$ with $N \in \mathbb{N}$, is $\sum_{j=1}^{N} j^2 \sim N^3$, cf. [47, 7.3].

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